ON SOME MODELS OF ACCEPTABLE RISK

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ABSTRACT

We consider discrete and continuous risk distribution functions. Acceptable risk distribution function is defined and different types of stochastic comparisons are discussed. Acceptable, unacceptable and intermediate regions for the level of loss are determined. The similar characterization is used for describing the loss for the outcomes in the sequence of harmful events. The loss is considered as acceptable, if either all events result in a loss from the acceptable region or not more than k of them result in a loss from the intermediate level. The Laplace transform methods are used for obtaining the probability of survival when harmful events from the Poisson process are 'too close' to each other.

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1. INTRODUCTION

Risk is usually understood as a danger that potentially harmful events represent to human beings, the environment or the economic values. Numerical outcomes of these events can be considered as realizations of a random variable C, which in concrete applications describe an economic loss, a number of casualties etc. Therefore, in this paper risk is measured by the corresponding loss and we will use these terms interchangeably. Usually (see, for example, Ushakov and Harrison, 1994, Vrijling, 1995), the discrete setting is described in terms of probabilities and outcomes as the following sequence:

$$(p_0, c_0 = 0), (p_1, c_1), (p_2, c_2), ..., (p_n, c_n), \tag{1}$$

where p_i is the probability of occurrence of realization i, whereas $c_i, 0 \le i \le n$ is the value of loss associated with this realization. Some of these realizations can be harmless, which means that the loss in this case is equal to 0. The corresponding probability is denoted by p_0. Without loss of generality, assume that c_i are strictly ordered:

$$c_0 = 0 < c_1 < c_2 < ... < c_n < \infty \ . \tag{2}$$

Therefore, the cumulative distribution function (Cdf) of risk $G(c)$ is simply defined via (1) and (2) as

$$G(c)=\begin{cases} p_0 = 1-\sum_1^n p_i, & 0\le c<c_1 \\ p_0+p_1, & c_1\le c<c_2 \\ \cdots\cdots\cdots\cdots & \\ \sum_0^{n-1} p_i = 1-p_n, & c_n\le c<c_n \\ 1, & c_n\le c<\infty \end{cases} \tag{3}$$

The expected value of loss is:

$$E[C]=\sum_0^n p_i c_i\,, \tag{4}$$

Relationship (4) can be already considered as some measure of risk, although usually it is too crude. It can be also used for comparison of different risks. Other relevant types of stochastic comparisons will be discussed in the next section.

Similar to the discrete case, the Cdf

$$G(c)=\Pr[C\le c]\,. \tag{5}$$

is the probability that the loss will not exceed $c: 0\le c<\infty$.

Closely related to the notion of risk is the notion of survivability. It can be interpreted in many ways. By survivability of some system we shall understand its ability to perform the required function without damage and loss or with damage or loss not exceeding some prescribed value. The probability of performing this function under stated conditions defines the corresponding measure of survivability

2. COMPARISON OF RISKS

As loss is usually inevitable, it is important to control and minimize it. At many instances a useful notion of acceptable loss c_a can be defined and the system's performance can be qualified as acceptable if, for instance:

$$E[C]\le c_a\ . \tag{6}$$

The value of c_a usually presents a rather natural requirement, i.e., the mean loss should be bounded by some reasonable amount. Ordering (6), however, is obviously not always sufficient because the large fluctuations of damage often cannot be accepted. The natural way to deal with this problem from the theoretical point of view is to consider the generalization of (6) via the acceptable risk function.

Let C_a denote a random variable of acceptable loss with $G_a(c)$ as its Cdf. It means that we accept the real loss C, if it is less in some reasonable stochastic sense than C_a. The ordering (comparison) of the means of the corresponding Cdfs is the simplest ordering of this type. Defining $G_a(c)$ is a much more complex task than defining c_a in (6), and it can need a detailed probabilistic analysis, case studies of similar systems or situations, expert's opinions etc. If C does not exceed C_a, then we assume that the real loss is acceptable. Thus, the main steps in the probabilistic risk analyses of the described type are: to obtain $G(c)$, to define $G_a(c)$ and to perform the corresponding stochastic comparison. In what follows in this section we consider some simple aspects of stochastic comparison for risks.

A well-known (Ross, 1996; Shaked and Shantikumar, 2006) and widely used type of stochastic comparison is the (usual) stochastic ordering defined as follows. If

$$G_a(c) \le G(c) \left(\overline{G}_a(c) \ge \overline{G}(c)\right), \forall c \in [0,\infty); \overline{G} \equiv 1 - G, \tag{7}$$

then

$$C \le_{st} C_a, \tag{8}$$

and the loss is considered as acceptable. Inequality (8) is a rather strong type of ordering. On the other hand, inequality (6) describes the weakest version of comparison. It is interesting to consider some relevant intermediate types of ordering taking into account 'variability' of C.

According to Ross (1996) (see also Kaas et al, 2001) a random variable C_a is said to be stochastically more variable than a random variable C, if for all $b \ge 0$:

$$\int_b^\infty \overline{G}_a(c)dc \ge \int_b^\infty \overline{G}(c)dc. \tag{9}$$

When $b = 0$, equation (11) reduces to the comparison of the means.

The Cdf $G_a(c)$ for the discrete case can be defined by equation (3), where p_i should be substituted by p_{ia}. If acceptable values of $p_{ia}, i = 1,2,...,n$ are obtained (this should be performed for each specific situation and can be based on various information including expert opinions etc.), then the obvious acceptance rule is:

$$p_i \le p_{ia}, \quad i = 1,2,...,n \tag{10}$$

and it is clearly seen from (3) that these inequalities lead to stochastic ordering (7). Thus, (10) is a rather convenient sufficient condition for this ordering. It is also clear that it is not a necessary condition, which can be illustrated by the following example.

Example 1

Let $n = 2$. Then

$$G(c) = \begin{cases} 1 - p_1 - p_2, & 0 \le c < c_1 \\ 1 - p_2, & c_1 \le c < c_2 \\ 1, & c_2 \le c < \infty \end{cases} \tag{11}$$

and $G_a(c)$ is obtained via substituting p_1, p_2 by $p_{1a}, p_{2,a}$, respectively. Inequality (7) holds for this case, if

$$\begin{aligned} p_1 + p_2 &\le p_{1a} + p_{2a} \\ p_2 &\le p_{2a} \end{aligned} \tag{12}$$

It is obvious that inequalities (10) for $n = 2$ imply (12) but not visa versa. On the other hand, the comparison in variability (9) leads to:

$$\begin{aligned} p_1c_1 + p_2c_2 &\le p_{1a}c_1 + p_{2a}c_2 \\ p_2 &\le p_{2a} \end{aligned} \tag{13}$$

As $c_2 > c_1$, inequalities (12) imply inequalities (13) but not visa versa. This fact illustrates for the specific case under consideration that the comparison in variability is intermediate between the stochastic comparison (7) and the comparison in the mean. The latter in this example is defined by the first inequality in (17).

Using probabilities p_{ia} and similar to (4), one can define the acceptable mean loss for the general discrete case as

$$c_a = E[C_a] = \sum_1^n p_{ia}c_i .$$

Thus, if $p_i \le p_{ia}$, then

$$E[C] \equiv \sum_1^n p_i c_i \le E[C_a] \equiv c_a . \tag{14}$$

The inverse is not necessarily true (as illustrated by Example 1): there can exist sequences of probabilities p_i for which inequality (14) is valid whereas inequality (10) does not hold.

3. RECURRENT EVENTS

3.1 The process of harmful shocks

In this section we will consider risks caused by recurrent events. Assume that a sequence of possibly harmful instantaneous events is modeled by a stochastic point process. Assume also for simplicity that it is a non-homogeneous Poisson process (NHPP) with rate $\lambda(t)$. For convenience, let us call these events shocks. As previously, each shock is causing a random loss of amount C_i. Let $C_i, i-1,2,...$ be i.i.d random variables with the continuous Cdf $G(c)$. Our interest is in considering overall consequences of shocks in $[0,t)$.

Divide the $c-$axis into n regions

$$(0,\hat{c}_1], (\hat{c}_1,\hat{c}_2],...,(\hat{c}_{n-1},\infty) . \tag{15}$$

The probability that the loss does not exceed level $\hat{c}_i$ is $G(\hat{c}_i)$ and the probability that it is in the region $(\hat{c}_i,\hat{c}_j], i < j; i, j < n; \hat{c}_n \equiv \infty$ is

$$p_{i,j} = G(\hat{c}_j) - G(\hat{c}_i), \quad p_{i,n} = 1 - G(\hat{c}_i), \quad p_i = G(\hat{c}_i) - G(0) = G(\hat{c}_i). \tag{16}$$

The first important step is to derive the probability $P_j(t)$ that all events that occurred in $(0,t]$ had resulted in a loss not exceeding $\hat{c}_i$. This probability can be defined as

$$P_i(t) = \exp\left\{ -\int_0^t (1-p_i)\lambda(x)dx \right\}, \tag{17}$$

which can be easily seen directly:

$$P_i(t) = \exp\{-\Lambda(t)\} \sum_{k=0}^{\infty} \frac{(\Lambda(t))^k}{k!} p_i^k = \exp\left\{ -\int_0^t (1-p_i)\lambda(x)dx \right\},$$

where $\Lambda(t) = \int_0^t \lambda(u)du$.

The corresponding proof for a more general case, when $p_i(t)$ are functions of time, can be found in Block et al (1985), Finkelstein (2003), Thomson (1988) to name a few. Similar to (17), the probability that all events had resulted in a loss in the range from c_i to c_j can be defined as

$$P_{i,j}(t) = \exp\left\{ -\int_0^t (1-p_{i,j})\lambda(x)dx \right\}. \tag{18}$$

Specifically, for the 3 regions:

$$P_s(t) = \exp\left\{ -\int_0^t (1-p_s)\lambda(x)dx \right\}, \quad P_{s,u}(t) = \exp\left\{ -\int_0^t (1-p_{s,u})\lambda(x)dx \right\} \tag{19}$$

$$P_u(t) = \exp\left\{ -\int_0^t (1-p_u)\lambda(x)dx \right\}, \tag{20}$$

where $P_s(t)$ is the probability that all events from the Poisson process in $[0,t)$ result in a 'safe loss'; $P_{s,u}(t)$ denotes the probability that all events result in a loss in $[c_s, c_u)$. Eventually, $P_u(t)$ denotes the supplementary (not having practical importance) probability that all events result in a loss in the region $[c_u, \infty)$.

Hence, the strongest criterion of the acceptable risk is when all events result in a loss from the first region. Then the performance of a system can be considered as acceptable. It is reasonable to consider a weaker version of the acceptance criterion allowing, for instance, not more than $k = 1,2,...$ events to result in a loss from the intermediate region $[c_s, c_u)$ (an event in $[c_u, \infty)$ is "not allowed" at all). Thus, we want to assess the probability $P_{s,k}(t)$ that all events result in a loss, not exceeding c_u whereas not more than k of them are allowed to result in a loss in $[c_s, c_u)$. Let, for simplicity, the initial process be the HPP with rate λ. Then it can be split into 3 independent Poisson processes with rates

$$\lambda p_s, \lambda p_{s,u}, \lambda p_u \ . \tag{21}$$

Due to our weakened acceptance risk criterion, the risk in $[0,t)$ is defined as unacceptable if at least one event from the process with rate λp_u will occur or if more than k events from the process with rate $\lambda p_{s,u}$ will occur. These considerations lead to the following equation for the probability of safe (with acceptable risk) performance:

$$P_{s,k}(t) = \exp\{-\lambda p_u t\}\exp\{-\lambda p_{s,u} t\}\sum_0^k \frac{(\lambda p_{s,u} t)^i}{i!} \ . \tag{22}$$

When there is no intermediate region: $c_u = c_s$, we arrive at

$$P_{s,0}(t) \equiv P_s(t) = \exp\{-\lambda p_u t\} = \exp\{-\lambda(1-p_s)t\}\ , \tag{23}$$

which coincides with the first equation in (19) for this specific case.

3.2. Other acceptance criteria

Let now an external shock's impact be described by binary random variables with outcomes "survived" or "not survived". The latter event for convenience will be called "failure" although it can represent accident, disaster etc. If each event from the Poisson process with rate $\lambda(t)$ is survived with probability p and is not survived with a complementary probability $1-p$, then, similar to (17), the survival probability (all shocks are survived) in $[0,t)$ is given by

$$P(t) = \exp\left\{-\int_0^t (1-p)\lambda(x)dx\right\}. \tag{24}$$

Consider now a different criterion of survival. Assume that shocks from the NHPP with rate $\lambda(t)$ are harmless, if they are rather rare. A failure of a system occurs only when shocks are 'too close' (a system did not recover from the consequences of a previous shock). Let the recovering time τ be random with the Cdf $R(t)$. The corresponding survival probability can be written as (Finkelstein, 2007)

$$\begin{aligned} P(t) &= \exp\left\{-\int_0^t \lambda(u)du\right\}\left(1+\int_0^t \lambda(u)du\right) \\ &+\int_0^t \lambda(x)\exp\left\{-\int_0^x \lambda(u)du\right\}\left[\int_0^{t-x} \lambda(y)\exp\left\{-\int_0^y \lambda(u)du\right\}R(y)\hat{P}(t-x-y)dy\right]dx, \end{aligned} \tag{25}$$

where the first term in the right hand side is the probability that there was not more than one shock in $[0,t)$ and the integrand defines the joint probability of the following events

-the first shock occurred in $[x, x+dx)$,
-the second shock occurred in $[x+y, x+y+dy)$,
-the time between two shocks y is sufficient for recovering (probability- $R(y)$),
-the system is functioning without failures in $[x+y, t)$.

By $\hat{P}(t)$ in (25) we denote the probability of system's functioning without failures in $[0,t)$ given that the first shock had occurred at $t=0$. Similar to (25):

$$\hat{P}(t) = \exp\left\{-\int_0^t \lambda(u)du\right\} + \int_0^t \lambda(x)\exp\left\{-\int_0^x \lambda(u)du\right\} R(x)\hat{P}(t-x)dx. \tag{26}$$

Equations (25) and (26) can be solved numerically. For the case of the homogeneous Poisson process with rate λ these equations can be easily solved via the Laplace transform (Finkelstein, 2007):

$$P(s) = \frac{s[1-\lambda R(s+\lambda)] - \lambda^2 R(s+\lambda) + 2\lambda}{(s+\lambda)^2[1-\lambda R(s+\lambda)]}, \tag{27}$$

where $P(s)$ and $R(s)$ denote Laplace transforms of $P(t)$ and $R(t)$, respectively.

Example 2. Let τ be exponentially distributed: $R(t) = 1-\exp\{-\mu t\}$. Inverting the Laplace transform (27) for this special case:

$$P(t) = A_1 \exp\{s_1 t\} + A_2 \exp\{s_2 t\}, \tag{28}$$

where s_1, s_2

$$s_{1,2} = \frac{-(2\lambda+\mu) \pm \sqrt{(2\lambda+\mu)^2 - 4\lambda^2}}{2}$$

and

$$A_1 = \frac{s_1 + 2\lambda + \mu}{s_1 - s_2}, \quad A_2 = -\frac{s_2 + 2\lambda + \mu}{s_1 - s_2}.$$

Another example of acceptance criterion is as follows. Assume that we have two types of shock processes. The first is the process of potentially harmful shocks with rate λ_1 and the second is the process of 'healing' events with rate λ_2. A failure occurs when at least two shocks of the first type occur in a row. A healing event that occurs after the harmful event neutralizes the consequences of the previous shock. Similar to (25)-(26), the survival probability $P(t)$ is obtained from the following equations:

$$P(t) = \exp\{-\lambda_1 t\} + \tilde{P}(t)$$

$$\tilde{P}(t) = \exp\{-\lambda_1 t)\lambda_1 t + \int_0^t \lambda_2 \exp\{-\lambda_2 x\}\exp\{-\lambda_1 x\} \int_0^{t-x} \lambda_1 \exp\{-\lambda_1\}\tilde{P}(t-x-y)dydx.$$

The interpretation of the terms in this equation is similar to our reasoning when obtaining (25)-(26). Applying the Laplace transform to the second equation results in

$$\tilde{P}(s) = \frac{\lambda_1(\lambda_1 + \lambda_2 + s)}{(\lambda_1 + \lambda_2 + s)(\lambda_1 + s)^2 - \lambda_1\lambda_2(\lambda_1 + s)}.$$

Finally,

$$P(s) = \frac{2\lambda_1 + \lambda_2 + s}{(\lambda_1 + \lambda_2 + s)(\lambda_1 + s) - \lambda_1\lambda_2}. \tag{29}$$

The inverse Laplace transform for (29) results in equation (28) with the same values of s_1 and s_2 . This is not surprising, as it can be easily seen that both settings are probabilistically equivalent. Using the Laplace transforms technique, some other criterions of acceptance can be also considered (Finkelstein and Zarudnij, 2002).

4. CONCLUDING REMARKS

Probabilistic risk assessment is usually rather complicated. The main problem is to find a suitable model that will give a possibility of a reasonable mathematical description and at the same time be real and practical. In this paper we have considered some simple approaches for defining and modeling characteristics of acceptable risk.

The mean acceptable loss is a natural acceptance criterion, but in many situations it is obviously not sufficient. By defining the Cdf of acceptable loss $G_a(c)$ one can solve the problem of classification of the real loss theoretically via a suitable stochastic comparison of $G(c)$ with $G_a(c)$. Defining $G_a(c)$ in practice is rather subjective and should be justified by the detailed analyses of the outcomes of harmful events under consideration.

The other way of describing $G_a(c)$ is to perform a discretization of the c-axis by considering regions, e.g., with acceptable loss, intermediate loss and unacceptable loss. Alternatively, binary models for acceptance criteria can be also considered. In Section 3.2 we deal with a specific model, when the shocks from the Poisson process are 'not allowed' to be too close.

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